Mengenang Jejak Sebagian Kecil Bangsa Indonesia Yang Pernah Mengikuti Ujian Sekolah Pada Masa Awal Kemerdekaan
UJIAN PENGHABISAN SEKOLAH MENENGAH TINGKAT ATAS TAHUN 1948

ILMU UKUR SUDUT DAN SEGITIGA (TRIGONOMETRI)

1. **HBS Negeri Belanda (Nederland) 1948**
   a. Dari sebuah ΔABC diberikan $R = r_a - r$. Buktikan bahwa $\alpha = 60°$.
   b. Jika $\alpha = 60°$ dan $\beta \geq \gamma$, berapa harga-harga batas dari $\beta - \gamma$? Buktikanlah bahwa $-\frac{1}{2} \leq p \leq \frac{1}{4}$, jika $\cos\beta\cos\gamma = p$.
   c. Jika $\alpha = 60°$ dan $\cos\beta\cos\gamma = \frac{1}{8}$,hitunglah dengan tidak mempergunakan daftar log, $\sin\beta\sin\gamma$.
   d. Buktikanlah dengan apa yang diberikan pada c, bahwa luas ΔABC sama dengan $\frac{5}{8}R^2\sqrt{3}$.

Solusi:

a. $R = r_a - r$

$$\frac{a}{2\sin\alpha} = s\tan\frac{1}{2}\alpha - (s-a)\tan\frac{1}{2}\alpha$$

$$\frac{a}{2\sin\alpha} = a\tan\frac{1}{2}\alpha$$

$$1 = 2\sin\alpha\tan\frac{1}{2}\alpha$$

$$1 = 4\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha\frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha}$$

$$1 = 4\sin^2\frac{1}{2}\alpha$$

$$1 = 2(1-\cos\alpha)$$

$$1 = 2 - 2\cos\alpha$$

$$\cos\alpha = \frac{1}{2}$$

$$\alpha = 60°$$

b. $\alpha + \beta + \gamma = 180°$

$$\beta - \gamma = 180° - \alpha - 2\gamma$$

$$\beta - \gamma = 180° - 60° - 2\gamma$$

$$\beta - \gamma = 120° - 2\gamma$$

$$0 \leq \beta - \gamma < 120°$$

$$\cos 120° \leq \cos(\beta - \gamma) \leq \cos 0$$

$$\cos 120° \le \cos\beta\cos\gamma + \sin\beta\sin\gamma \le \cos 0$$

$$-\frac{1}{2} \le \cos\beta\cos\gamma + \sin\beta\sin\gamma \le 1$$

$$-\frac{1}{2} - \sin\beta\sin\gamma \le \cos\beta\cos\gamma \le 1 - \sin\beta\sin\gamma$$

$$-\frac{1}{2} \le \cos\beta\cos\gamma \le \frac{1}{4}$$

$$-\frac{1}{2} \le p \le \frac{1}{4}$$

c. $\alpha + \beta + \gamma = 180°$

$$\beta + \gamma = 180° - \alpha$$

$$\beta + \gamma = 180° - 60°$$

$$\beta + \gamma = 120°$$

$$\cos(\beta + \gamma) = \cos 120°$$

$$\cos\beta\cos\gamma - \sin\beta\sin\gamma = -\frac{1}{2}$$

$$\sin\beta\sin\gamma = \cos\beta\cos\gamma + \frac{1}{2}$$

$$\sin\beta\sin\gamma = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

d. Luas $\Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}(2R\sin B)(2R\sin C)\sin 60° = 2R^2\sin\beta\sin\gamma\left(\frac{1}{2}\sqrt{3}\right) = \frac{5}{8}R^2\sqrt{3}$

2. **HBS Negeri Belanda (Nederland) 1948**

Hitunglah persamaan-persamaan yang tersebut di bawah ini.

a. $24\sin x\cos x + 7 = 6(\text{tg}\, x + \text{cotg}\, x)$;

b. $6\sin^2 x - 4 = 3\sin x\cos x$

c. $\sin 3x = \cos x$

Solusi:

a. $24\sin x\cos x + 7 = 6(\text{tg}\, x + \text{cotg}\, x)$

$$24\sin x\cos x + 7 = 6\left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\right)$$

$$24\sin x\cos x + 7 = 6\left(\frac{\sin^2 x + \cos^2 x}{\sin x\cos x}\right)$$

$$12\sin 2x + 7 = 12\left(\frac{1}{\sin 2x}\right)$$

$$12\sin^2 2x + 7\sin 2x = 12$$

$$12\sin^2 2x + 7\sin 2x - 12 = 0$$

$$(4\sin 2x - 3)(3\sin 2x + 4) = 0$$

$\sin 2x = \frac{3}{4} = 0,75$ (diterima) atau $\sin 2x = -\frac{4}{3} = -1\frac{1}{3}$ (ditolak)

$2x = 48°35' + k \times 360°$ atau $2x = 180° - 48°35' + k \times 360°$

$x = 24°16' + k \times 180°$ atau $x = 65°42' + k \times 180°$

b. $6\sin^2 x - 4 = 3\sin x\cos x$

$3\left(2\sin^2 x\right) - 4 = \frac{3}{2}\left(2\sin x\cos x\right)$

$3\left(1-\cos 2x\right) - 4 = \frac{3}{2}\sin 2x$

$3 - 3\cos 2x - 4 = \frac{3}{2}\sin 2x$

$-6\cos 2x - 2 = 3\sin 2x$

$3\sin 2x + 6\cos 2x + 2 = 0$

$\sin 2x + 2\cos 2x = -\frac{2}{3}$

Ambillah $\tan\theta = 2 \Leftrightarrow \theta = 63°26'$

$\sin 2x + \tan\theta\cos 2x = -\frac{2}{3}$

$\sin 2x\cos\theta + \cos 2x\sin\theta = -\frac{2}{3}\cos\theta$

$\sin\left(2x+\theta\right) = -\frac{2}{3}\cos\theta$

$\sin-\left(2x+\theta\right) = \frac{2}{3}\cos\theta$

$\sin-\left(2x+63°26'\right) = \frac{2}{3}\cos 63°26' = 0,2982$

$\sin-\left(2x+63°26'\right) = \sin 17°21'$

$2x + 63°26' = -17°21' + k\times 360°$ atau $2x + 63°26' = -162°39' + k\times 360°$

$2x + 63°26' = 197°21' + k\times 360°$ atau $2x + 63°26' = 342°39' + k\times 360°$

$x = 66°55' + k\times 180°$ atau $x = 139°37' + k\times 180°$

c. $\sin 3x = \cos x$

$\sin 3x = \sin\left(90° - x\right)$

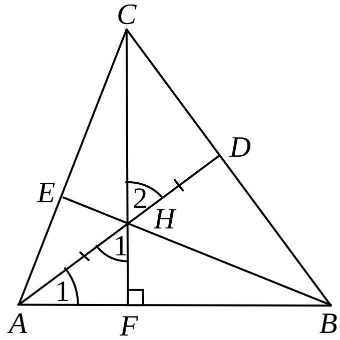
$3x = \left(90° - x\right) + k\times 360°$ atau $3x = \left(90° + x\right) + k\times 360°$

$x = 22°30' + k\times 180°$ atau $x = 45° + k\times 180°$

3. **Gymnasium Negeri Belanda, 1948**

Garis-garis tinggi AD dan BE dari ΔABC dibagi oleh titik tinggi H begitu rupa sehingga $AH = HD$ dan $BH = 4HE$. Hitunglah sudut-sudut ΔABC.

Solusi:



Pada ΔAHF diperoleh $\angle H_1 = 90° - \angle A_1$ .... (1)

Pada ΔABD diperoleh $\angle B = 90° - \angle A_1$ .... (2)

Dari persamaan (1) dan (2) diperoleh $\angle H_1 = \angle B = \beta$.

Pada ΔACF diperoleh $\cos\alpha = \frac{AE}{b} \Leftrightarrow AE = b\cos\alpha$ .... (3)

Pada ΔAHF diperoleh $\cot\angle H_1 = \frac{HF}{AF}$

$HF = AF\cot\angle H_1 = b\cos\alpha\cot\beta$

$HF = 2R\sin\beta\cos\alpha\frac{\cos\beta}{\sin\beta} = 2R\cos\alpha\cos\beta$ .... (4)

Pada ΔCHD diperoleh $\sin\angle H_2 = \frac{CD}{CH}$, sehingga

$$CH = \frac{CD}{\sin\angle H_2} = \frac{b\cos\gamma}{\sin\beta} = \frac{2R\sin\beta\cos\gamma}{\sin\beta} = 2R\cos\gamma \text{ .... (5)}$$

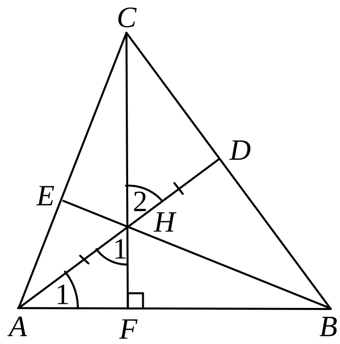
$$\cos\angle H_2 = \frac{HD}{CH} \to HD = CH\cos\angle H_2 = 2R\cos\gamma\cos\beta \text{ .... (6)}$$

$$\cos\angle H_1 = \frac{HF}{AH} \to AH = \frac{HF}{\cos\angle H_1} = \frac{2R\cos\alpha\cos\beta}{\cos\beta} = 2R\cos\alpha \text{ .... (7)}$$

$HD = AH$

$2R\cos\gamma\cos\beta = 2R\cos\alpha$

$\cos\gamma\cos\beta = \cos\alpha$ .... (8)



Pada ΔBHE diperoleh $\angle BHF = 90° - \angle FBH$ .... (9)

Pada ΔBAE diperoleh $\angle BAE = 90° - \angle FBH$ .... (10)

Dari persamaan (3) dan (4) diperoleh $\angle BHF = \angle BAF = \alpha$.

Pada ΔBAD diperoleh $\cos\beta = \frac{BD}{c} \Leftrightarrow BD = c\cos\beta$ .... (11)

Pada ΔBHF diperoleh $\cot\angle BHF = \frac{HF}{FB}$

$HF = FB\cot\angle BHF = a\cos\beta\cot\alpha$

$$HF = 2R\sin\alpha\cos\beta\frac{\cos\alpha}{\sin\alpha} = 2R\cos\beta\cos\alpha \text{ .... (12)}$$

Pada ΔCHE diperoleh $\cos\angle CHE = \frac{HE}{CH}$, sehingga

$HE = CH\cos\angle CHE = 2R\cos\gamma\cos\alpha$ .... (13)

$$\cos\angle BHF = \frac{HF}{HB} \to HB = \frac{HF}{\cos\angle BHF} = \frac{2R\cos\beta\cos\alpha}{\cos\alpha} = 2R\cos\beta \text{ .... (14)}$$

$BH = 4HE$

$2R\cos\beta = 4\cdot 2R\cos\gamma\cos\alpha$

$\cos\beta = 4\cos\gamma\cos\alpha$ .... (15)

Dari persamaan (8) dan (15) diperoleh:

$\cos\beta = 4\cos\gamma\cos\gamma\cos\beta$

$1 = 4\cos^2\gamma$

$\cos^2\gamma = \frac{1}{4}$

$\cos\gamma = \frac{1}{2} \vee \cos\gamma = -\frac{1}{2}$

$\gamma = 60° \vee \gamma = 120°$

$\gamma = 120° \to \cos\gamma\cos\beta = \cos\alpha$

$\cos 60°\cos\beta = \cos\alpha$

$\frac{1}{2}\cos\beta = \cos\left[180° - (\beta+\gamma)\right]$

$\frac{1}{2}\cos\beta = \cos\left[180° - (\beta+60°)\right]$

$$\frac{1}{2}\cos\beta = \cos\left(120^\circ - \beta\right)$$

$$\frac{1}{2}\cos\beta = \cos 120^\circ \cos\beta + \sin 120^\circ \sin\beta$$

$$\cos\beta = -\cos\beta + \sqrt{3}\sin\beta$$

$$2\cos\beta = \sqrt{3}\sin\beta$$

$$\tan\beta = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = 1{,}1547 \to \beta = 49{,}11^\circ$$

$$\alpha = 180^\circ - \left(\beta + \gamma\right) = 180^\circ - \left(49{,}11 + 60\right)^\circ = 70{,}89^\circ$$

$$\gamma = 120^\circ \to \cos\gamma\cos\beta = \cos\alpha$$

$$\cos 120^\circ \cos\beta = \cos\alpha$$

$$-\frac{1}{2}\cos\beta = \cos\left[180^\circ - \left(\beta + \gamma\right)\right]$$

$$-\frac{1}{2}\cos\beta = \cos\left[180^\circ - \left(\beta + 120^\circ\right)\right]$$

$$-\frac{1}{2}\cos\beta = \cos\left(60^\circ - \beta\right)$$

$$-\frac{1}{2}\cos\beta = \cos 60^\circ \cos\beta + \sin 60^\circ \sin\beta$$

$$\cos\beta = -\cos\beta - \sqrt{3}\sin\beta$$

$$2\cos\beta = -\sqrt{3}\sin\beta$$

$$\tan\beta = -\frac{2}{\sqrt{3}} = -\frac{2}{3}\sqrt{3} = -1{,}1547 \to \beta = 130{,}89^\circ \text{ (ditolak)}$$